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The Mathematics of Electro-Chemical Machining

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1. INTRODUCTION

Electro-chemical machining is an extremely efficient method of shaping very hard metals by means of electrolysis. The metal to be shaped is the anode and an electrostatic potential set up between it and the cathode causes the metal of the anode to be worn away at a rate proportional to the current density which in turn depends on the electrostatic field. The liquid electrolyte is driven through the space between anode and cathode to wash away the displaced material. Under the simplest assumptions, which are those made here, the mathematical problem is that of determining an unknown "free" surface, that of the anode, when the shape and motion of the cathode are given. This is closely related to the problem of determining free surfaces in hydrodynamics and these similarities are used in the ensuing analysis. It is very desirable to explore the mathematical aspects of this problem as fully as possible in order to suggest the most efficient shape of cathode design and to minimise the amount of metal lost in a given machining process.

The object of the present paper is first to formulate the boundary value problem in a very general way and then to put this in a form which enables us to draw on well known results from hydrodynamics and potential theory. This is carried out in Section 2. In Section 3 the one-dimensional case is completely analysed. In Section 4 some problems using self-similar variables are discussed and in Section 5 the inverse problem of finding the cathode when the anode is prescribed is solved for the quasi-stationary case in two dimensions. This leads to a discussion of the extent to which the problem is well-posed, the significance of the characteristics in an extended problem obtained by analytic continuation, and consideration of the corresponding three-dimensional problem with axial symmetry.

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2. Formulation

For simplicity we formulate the general problem in two dimensions but the corresponding equations in three dimensions are easily obtained. We assume that the equation of the anode is $y = a(x, t)$ and that the electrostatic potential is $\phi' = 0$ there while the cathode has equation $y = c(x, t)$ and is at constant potential $\phi' = -\phi'_0$, where $\phi'_0 > 0$. In $a(x, t) < y < c(x, t)$ the potential $\phi'(x, y, t)$ is harmonic in x and y and the electric field causes erosion at the anode according to the law

$$\frac{d\mathbf{r}}{dt} = M\nabla\phi',$$

where M is a positive constant and $\mathbf{r}(t) = (x(t), y(t))$ is a point on the anode at time t. Thus

$$\mathbf{r}(t + \delta t) = \mathbf{r}(t) + M\nabla\phi'\delta t$$

and, since $y(t + \delta t) = a(x(t + \delta t),\ t + \delta t)$, we proceed to obtain the condition

$$\frac{\partial a}{\partial t} + M\frac{\partial \phi'}{\partial x}\frac{\partial a}{\partial x} = M\frac{\partial \phi'}{\partial y}$$

on $y = a(x, t)$. We now set $\phi = M\phi'$, $\phi_0 = M\phi'_0$ to obtain the boundary value problem

$$\nabla^2\phi = 0 \qquad \text{in} \quad a(x, t) < y < c(x, t), \tag{1}$$

$$\phi = -\phi_0 \qquad \text{on} \quad y = c(x, t), \tag{2}$$

$$\phi = 0 \qquad \text{on} \quad y = a(x, t), \tag{3}$$

$$\frac{\partial a}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial a}{\partial x} = \frac{\partial \phi}{\partial y} \qquad \text{on} \quad y = a(x, t). \tag{4}$$

This is illustrated in Fig. 1.

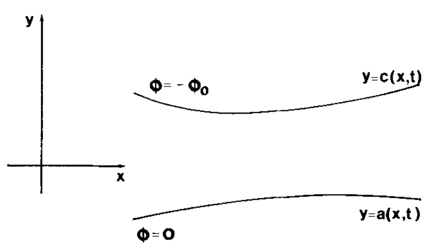

FIGURE 1

Throughout this paper the shape and motion of the cathode are described by variants of the letter c and of the anode by corresponding variants of the letter a.

The natural problem to consider is that of determining the anode shape when the shape and motion of the cathode, which is the machining tool, are known. In practice the cathode will normally have a fixed shape but the more general possibility is easily accommodated. Thus we assume that $c(x, t)$ is given for $t \geqslant 0$ and that the initial position $a(x, 0)$ of the anode is also given. The problem is then to determine the subsequent shape of the anode, that is, to find $a(x, t)$ when $t > 0$.

Equations (1)–(4) are very similar to those which occur in free surface problems in hydrodynamics and the electrostatic potential ϕ' has been deliberately scaled to one with the dimensions of a hydrodynamic potential to take advantage of this. In the inviscid or classical theory, condition (3) would normally be replaced by a condition of constant pressure. On the other hand, in the model of slow viscous flow associated with the Hele-Shaw cell both conditions (3) and (4) are precisely those commonly taken at the free surface of the viscous liquid.

3. The Case of Plane Electrodes

When the electrodes are plane and parallel to the x-axis, the general problem can readily be solved. Although the mathematics is straightforward, we give a brief presentation as there are some features of interest which relate to later work. Equations (1)–(4) now become

$$\frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{in } a(t) < y < c(t) \quad \text{where} \quad \phi = \phi(y, t), \tag{5}$$

$$\phi = -\phi_0 \qquad \text{on} \quad y = c(t), \tag{6}$$

$$\phi = 0 \qquad \text{on} \quad y = a(t), \tag{7}$$

$$\frac{da}{dt} = \frac{\partial \phi}{\partial y} \qquad \text{on } y = a(t). \tag{8}$$

From (5), (6), and (7)

$$\phi = \frac{\phi_0(y - a(t))}{a(t) - c(t)}$$

and then from (8)

$$\{a(t) - c(t)\}\, a'(t) = \phi_0 \tag{9}$$

which can also be written as

$$s'(t) = \frac{\phi_0}{s(t)} + c'(t), \tag{10}$$

where $s(t) = c(t) - a(t) > 0$ is the separation between the two plates. It is clear from (10) that s can never vanish since s' becomes positive as $s \to 0$. With $c(t)$ and $a(0)$ given Eqs. (9) or (10) can be solved. A quasi-stationary solution corresponding to a constant value of s, say $s = l$, will occur if the cathode moves with constant speed V in the negative y-direction where $\phi_0 = Vl$.

If the cathode is moved at this constant speed so that $c'(t) = -V$ but the initial separation is $s_0 \neq l$, then (10) can be integrated explicitly to give

$$Vt + s - s_0 = l \ln \frac{l - s_0}{l - s}. \tag{11}$$

This shows that a quasi-stationary solution $s = l$ is assumed in the limit as $t \to \infty$. Moreover it is approached exponentially quickly since for large t

$$|s - l| \sim l e^{-Vt/l}.$$

We also note that when the cathode is held stationary for all $t \geqslant 0$

$$s(t) = (2\phi_0 t + s_0^2)^{1/2}. \tag{12}$$

These solutions can be used as a basis for a linear perturbation by assuming an anode shape $a(x, t) = a(t) + \varepsilon A(x, t)$ for small ε, thereby giving rise to classical boundary value problems. Such an analysis has been carried out by Fitz-Gerald and McGeough [1], who assume an initial roughness on the anode and obtain estimates for the rate at which this will be smoothed out in the machining process. Smoothing is to be expected since at those parts of the anode which are near the cathode the electric field is stronger.

4. Self-Similar Solutions

There would in general be a simplification of the equations if the number of independent variables were reduced from three to two by making use of similarity principles. If we look at Eqs. (1)–(4), the only prescribed dimensional quantity that appears is the constant ϕ_0 which has dimension L^2T^{-1}. If then we were to formulate a problem which could be specified by using constants of dimension L^2T^{-1} only, we would expect a similarity solution. An appropriate configuration is shown in Fig. 2 where at $t = 0$ the cathode

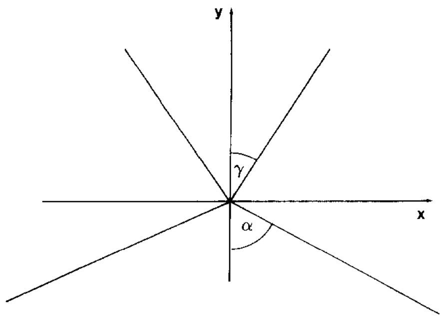

FIGURE 2

is the wedge $y = \cot\gamma\,|x|$ and the anode is the wedge $y = -\cot\alpha\,|x|$. If for $t > 0$ the cathode moves so that its tip has the trajectory $y = -c\phi_0^{1/2} t^{1/2}$, where c is dimensionless, we will have a self-similar solution. In terms of dimensionless variables

$$\bar{x} = \frac{x}{t^{1/2}\phi_0^{1/2}}, \qquad \bar{y} = \frac{y}{t^{1/2}\phi_0^{1/2}}, \qquad \bar{\phi} = \frac{\phi}{\phi_0},$$

Eqs. (1)–(4) reduce to

$$\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} = 0 \qquad \text{in} \quad \bar{a}(\bar{x}) < \bar{y} < \bar{c}(\bar{x}), \tag{13}$$

$$\bar{\phi} = -1 \qquad \text{on} \quad \bar{y} = \bar{c}(\bar{x}), \tag{14}$$

$$\bar{\phi} = 0 \qquad \text{on} \quad \bar{y} = \bar{a}(\bar{x}), \tag{15}$$

$$\frac{\partial\bar{\phi}}{\partial\bar{y}} - \tfrac{1}{2}\bar{y} = \bar{a}'(\bar{x})\left(\frac{\partial\bar{\phi}}{\partial\bar{x}} - \tfrac{1}{2}\bar{x}\right) \qquad \text{on} \quad \bar{y} = \bar{a}(\bar{x}). \tag{16}$$

Here $\bar{c}(\bar{x}) = \cot\gamma|\bar{x}| - c$ and $\bar{y} = \bar{a}(\bar{x})$ is the anode shape in the dimensionless variables which is unknown except that $\bar{a}(\bar{x}) \sim -|\bar{x}|\cot\alpha$ for large $|\bar{x}|$. For a plane cathode $\gamma = \pi/2$ and the problem is illustrated in Fig. 3 where the shape of the unknown curve $\bar{y} = \bar{a}(\bar{x})$ has to be determined from the otherwise over-specified boundary value problem. This technique has been widely used in fluid mechanics, notably in the water entry problem where a solid wedge is plunged at constant speed into a half-plane of incompressible liquid and the splash contour has to be found (see, for example, [2, 5, 9]). The problem illustrated in Fig. 3 would seem to be more analytically tractable because the awkward singularity which occurs at the interface of the solid and the liquid on the wedge side in the water entry problem does not arise here.

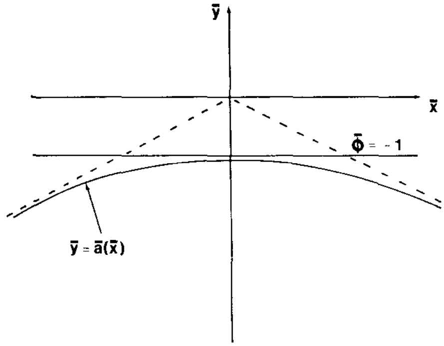

FIGURE 3

When $\alpha = \gamma = \pi/2$ we can easily solve (13)–(16) since $\bar{\phi}$ is a function of $\bar{y}$ only, $\bar{c}(\bar{x}) = -c$ and $\bar{a}(\bar{x}) = -a$, where a is a constant. The potential $\bar{\phi}$ is given by

$$\bar{\phi} = \frac{\bar{y} - a}{a - c}$$

and from Eq. (16) we get $a(a-c) = 2$ or

$$a = \frac{c + (c^2 + 8)^{1/2}}{2} \tag{17}$$

which gives the rate at which the anode moves down when the cathode moves with prescribed $t^{1/2}$ dependence. This result could, of course, have been obtained directly from (9). In fact the solution (12) when $s_0 = 0$ is the special case of (17) when $c = 0$.

We can generalise this result by assuming that the potential difference is time-dependent. If, for example, $\phi_0 = kt^n$, then k has dimension L^2T^{-n-1}. When $n = 1$ this is L^2T^{-2} and we can consider the case $\phi_0 = -U^2t$ where U is some velocity. The configurations of Figs. 2 and 3 hold as before but now the cathode must move down with constant speed V, say. If we now set $\bar{x} = x/Ut$, $\bar{y} = y/Ut$, $\bar{\phi} = \phi/U^2t$, and $c = V/U$ we get precisely the same boundary value problem as before except that (16) is now replaced by

$$\frac{\partial \bar{\phi}}{\partial \bar{y}} - \bar{y} = \bar{a}'(\bar{x}) \left(\frac{\partial \bar{\phi}}{\partial \bar{x}} - \bar{x} \right) \qquad \text{on} \quad \bar{y} = \bar{a}(\bar{x}).$$

This latter condition is identical with that which holds on the free surface in dimensionless variables in the water entry problem.

Finally, we observe that, although the wedges in Fig. 2 are symmetric with respect to the y-axis, the same analysis could be applied to non-symmetric wedges so long as both tips are at the origin. Also the tip of the cathode need not move down the y-axis but it must move in a straight line with the appropriate time dependence since any curved path would introduce an unwanted constant with the dimension of length.

5. Quasi-stationary Solutions

A special case of the boundary value problem (1)–(4) with important practical applications arises when both anode and cathode are moving in the negative y-direction with constant speed V and the solution is independent of time when formulated in terms of coordinates fixed in the cathode (or anode). The appropriate equations, obtained by replacing $a(x, t)$ by $a(x) - Vt$, $c(x, t)$ by $c(x) - Vt$, are

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \text{in} \quad a(x) < y < c(x), \tag{18}$$

$$\phi = -\phi_0 \qquad \text{on} \quad y = c(x), \tag{19}$$

$$\phi = 0 \qquad \text{on} \quad y = a(x), \tag{20}$$

$$-V + a'(x)\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} \qquad \text{on} \quad y = a(x). \tag{21}$$

When combined with (20), the boundary condition (21) can be written in the alternative form

$$\frac{\partial\phi}{\partial n} = V\cos\beta \qquad \text{on} \quad y = a(x), \tag{22}$$

where $\partial/\partial n$ represents differentiation along the normal into the region $y < a(x)$ and β is the angle which this normal makes with the negative y-axis. When $a(x)$, $c(x)$ are constants a, c respectively and $l = c - a$ we immediately recover the elementary plane solution obtained in Section 3 with $\phi_0 = Vl$.

Investigation of a linear perturbation of this solution provides a helpful indication of what is likely to happen in the non-linear case. If the equation of the cathode is $y = l + \varepsilon\tilde{c}(x)$, the equation of the anode is $y = \varepsilon\tilde{a}(x)$ and we set

$$\phi = \frac{\phi_0}{l}(-y + \varepsilon\tilde{\phi}),$$

then, to first order in ε, the boundary value problem for $\tilde{\phi}$ is

$$\frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} = 0 \qquad \text{in} \quad 0 < y < l, \tag{23}$$

$$\tilde{\phi} = \tilde{c}(x) \qquad \text{on} \quad y = l, \tag{24}$$

$$\tilde{\phi} = \tilde{a}(x) \qquad \text{on} \quad y = 0, \tag{25}$$

$$\frac{\partial \tilde{\phi}}{\partial y} = 0 \qquad \text{on} \quad y = 0. \tag{26}$$

Equations (23), (24), and (26) together constitute a well-posed problem for $\tilde{\phi}$, the solution to which is best obtained for our purposes by continuing $\tilde{\phi}$ as an even function across $y = 0$, replacing (26) by the condition $\tilde{\phi} = \tilde{c}(x)$ on $y = -l$, and solving the Dirichlet problem in the strip $-l < y < l$. This can be done by a variety of methods and when we set $y = 0$ in the resulting solution to recover $\tilde{a}(x)$ through (25), we obtain

$$\tilde{a}(x) = \frac{1}{2l} \int_{-\infty}^{\infty} \tilde{c}(\theta + x) \operatorname{sech} \frac{\pi\theta}{2l} \, d\theta. \tag{27}$$

The advantage of solving the problem in this way is that it highlights the smoothness of the resulting anode shape. As an equipotential in the interior of the domain, the curve $y = \tilde{a}(x)$ is analytic even if $c(x)$ is only continuous. The maximum principle can also be invoked to establish that $\max |\tilde{a}(x)| < \max |\tilde{c}(x)|$. From (27) the actual shape of the anode can easily be found by numerical integration when the cathode is given.

Up to now we have thought of the position and shape of the cathode as known and have regarded the position of the anode as the unknown to be found. The inverse problem of finding the cathode given the anode is physically important and is of considerable mathematical interest. A metal has often got to be shaped to a prescribed curve and this gives rise to the design problem of making the cathode the correct shape to ensure this. In the quasi-stationary case discussed in this section, this means that $a(x)$ is given and that Eqs. (20) and (22) constitute Cauchy data which determine a unique ϕ in the neighbourhood of $y = a(x)$ if $a(x)$ is an analytic function of x. The curve defined by $\phi = -\phi_0$ in the ensuing solution, or indeed any other appropriate equipotential, is then a possible cathode shape.

A very similar problem was considered by Saffman and Taylor [8] but in the quite different physical context of viscous flow in Hele-Shaw cells. In the context of electro-chemical machining it was discussed by Krylov [4] and later by Nilson and Tsuei [6, 7], who essentially used the same method as Saffman and Taylor. We use a somewhat different method here which enables us to discuss more fully the limitations which arise from the fact that the Cauchy problem for Laplace's equation is not in general well-posed and to obtain some comparable results for the corresponding problem in three dimensions with axial symmetry.

We assume that $y=a(x)$ is a given analytic function of x and use the conformal mapping

$$z = Z + ia(Z) \tag{28}$$

to map the curve $y=a(x)$ into the real axis $y=0$ of the X, Y-plane where $Z=X+iY$. Now

$$\left[-\frac{\partial\phi}{\partial Y}\right]_{Y=0} = \left[\frac{\partial\phi}{\partial n}\right]_{y=a(x)} \cdot \left|\frac{dz}{dZ}\right|_{Y=0} \tag{29}$$

where we have used standard properties of conformal mappings. Using (22) and (28) we see that the right-hand side of (29) is

$$V\cos\beta\,|1+ia'(X)| = \frac{V(1+a'(x)^2)^{1/2}}{V(1+a'(x)^2)^{1/2}} = V.$$

Since $\phi=0$ on $Y=0$ from (20) we obtain the solution of the potential problem in the X, Y-plane as $\phi=-VY$. The cathode is then given by setting $\phi=-\phi_0$ or $Y=\phi_0/V$. Equation (28) with $Y=\phi_0/V$ then gives the equation of the cathode in the original x, y-plane with X as parameter.

A slight generalisation is possible when the curve $y=a(x)$ is given parametrically, say by $x=x_0(\theta)$, $y=y_0(\theta)$ where these are analytic functions of θ. The appropriate mapping is now $z=x_0(Z)+iy_0(Z)$ and the boundary conditions in the X, Y-plane are

$$\phi=0, \qquad \frac{\partial\phi}{\partial Y} = -Vx_0'(X) \qquad \text{when} \quad Y=0. \tag{30}$$

As a first example we consider the anode to be the parabola $y=x^2/L$. Then

$$z = Z + \frac{iZ^2}{L} \tag{31}$$

and the cathode is given by

$$\begin{aligned} x &= (1-2\lambda)\,X, \\ y &= (\lambda-\lambda^2)\,L + X^2/L, \end{aligned} \tag{32}$$

where

$$\lambda = \frac{\phi_0}{VL}. \tag{33}$$

This is a parabola with the same axis as the original. Inspection of (32) suggests that λ should be restricted to the range $0 < \lambda \leqslant \frac{1}{2}$. We will discuss this more fully later. We note, as did Nilson and Tsuei, that when $\lambda = \frac{1}{2}$ the cathode is just a line (in practice a plane whose normal is parallel to Ox). For the given anode with fixed L we therefore have a family of possible cathodes. We can choose both ϕ_0 and V subject to the restriction $\phi_0 \leqslant 2VL$.

A practical problem which sometimes arises in electro-chemical machining is that of creating an indentation with high curvature. A shaped cathode is essential for this since the effect of using a plane cathode is one of smoothing as noted earlier at the end of Section 3. We seek the shape of the cathode which will give a high curvature indentation by exploring the case when the anode is the rectangular hyperbola $y^2 - x^2 = L^2$ $(L > 0)$ and letting $L \to 0$. The required mapping is

$$z = L \sinh Z + iL \cosh Z \tag{34}$$

and, using (30), we obtain the solution

$$\phi = -LV \cosh X \sin Y$$

in the X, Y-plane. Setting $\phi = -\phi_0$ we find that the equation of the cathode is

$$x = L \sinh X(\cos Y - \sin Y), \tag{35}$$

$$y = L \cosh X(\cos Y + \sin Y), \tag{36}$$

$$\lambda = \cosh X \sin Y, \tag{37}$$

with λ defined as before by (33). We cannot obtain the form of the cathode explicitly but all the necessary information about it is contained in parametric form through Eqs. (35)–(37). As in the case of the parabola we wish $x \geqslant 0$ when $X \geqslant 0$ and therefore require $Y \leqslant \pi/4$ which from (37) means that $\lambda \leqslant 1/\sqrt{2}$. The anode is seen to be symmetric about the y-axis as expected. The coordinates of the tip are $(0, L(\lambda + \sqrt{1-\lambda^2}))$ and the radius of curvature there is found, after some algebra, to be

$$\rho = L(\sqrt{1-\lambda^2} - 2\lambda + 2\lambda^3)$$

which is zero at the limiting case $\lambda = 1/\sqrt{2}$. Thus in practice we cannot let

L, which is the radius of curvature of the anode when $x=0$, tend to zero since the restriction $\lambda \leqslant 1/\sqrt{2}$ requires that

$$L \geqslant \sqrt{2}\phi_0/V.$$

We must now look more closely at why these restrictive inequalities $\lambda \leqslant \frac{1}{2}$ and $\lambda \leqslant 1/\sqrt{2}$ exist in these two examples, respectively. One obvious place to expect difficulties is at any singularities of the transformation (28), that is, at points where $dz/dZ=0$. From (31) such a point occurs for the parabola when $Z=iL/2$ corresponding to $x=0$, $y=L/4$ which is the tip of the critical cathode given by $\lambda=\frac{1}{2}$. Likewise from (34) for the hyperbola $dz/dZ=0$ when $\tanh Z=i$ corresponding to $x=0$, $y=\sqrt{2}L$ which is the tip of the critical cathode given by $\lambda=1/\sqrt{2}$ where the curvature is infinite.

To leave the matter there, however, would not be entirely satisfactory. In the first place it is not absolutely clear that the singularity is endemic to the problem itself and not caused by the particular method of solution. Second, and more important, it gives no clue as to whether or in what way the presence of these singularities is associated with the attempt to solve a Cauchy problem for an elliptic partial differential equation. To obtain a deeper insight into these matters it is necessary to consider the analytic continuation of the solution in the manner suggested in the last chapter of Garabedian [3] and this we now proceed to do.

We keep y as a real variable but regard x as a complex variable $\xi+i\eta$. The solution of the Cauchy problem (18), (20), and (22) is an analytic function of x and can be thought of as a complex function of three real variables ξ, η, and y (Fig. 4). We illustrate with the case of the hyperbola discussed above. In the plane $\eta=0$ the boundary value problem is precisely that already solved in the x, y-plane. In the plane $\xi=0$, however, we have the problem of determining the solution of

$$\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial \eta^2}=0 \tag{38}$$

which takes given values of ϕ, $\partial\phi/\partial n$ on the circle $\xi^2+\eta^2=L^2$. Now Eq. (38) is hyperbolic and the Cauchy problem is well-posed in general. However, at points on the data curve where the tangent has slope ± 1 which are the slopes of the characteristics $y\pm\eta=\text{constant}$, a singularity such as a branch point may be expected to occur and will propagate along the associated characteristic. Such a point is $y=L/\sqrt{2}$, $\eta=L/\sqrt{2}$ and the characteristic through this point meets the ξ, y-plane at $\xi=0$, $y=\sqrt{2}L$.

We can generalise this result before discussing its significance. If $a(z)$ is an even analytic function of z, say $b(z^2)$, then the curve in the y, η-plane on which Cauchy data will be assigned is $y=b(-\eta^2)$. The slope of the tangent

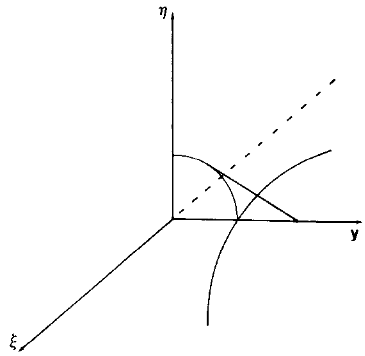

FIGURE 4

to this curve at the point $(\eta_0, b(-\eta_0^2))$ will be -1 if η_0 is a root of the equation

$$2\eta b'(-\eta^2) = 1 \tag{39}$$

and the tangent at this point will meet the y-axis where $y = \eta_0 + b(-\eta_0^2)$.

If we now consider the singular points of the transformation $z = Z + ib(Z^2)$ which is equivalent to (28) we see that these occur when $2Zb'(Z^2) = i$, that is, at $Z = iY$ where $2Yb'(-Y^2) = 1$ which means that $Y = \eta_0$ by (39). The singular point in the z-plane is thus $x = 0$, $y = \eta_0 + b(-\eta_0^2)$ in agreement with the characteristics argument.

By performing the analytic continuation of the problem in the large through the above analysis we are able to identify the singularities which occur when the solution of the Cauchy problem for Laplace's equation is sought. Moreover the resulting geometrical argument is also applicable to the corresponding problem with axial symmetry where x is replaced by r, the distance of a point from the y-axis. The transformation (28), with r replacing x, can be used as before and we can again explore the analytic continuation of the boundary value problem by now setting $r = \xi + i\eta$. The differential equation in the plane $\xi = 0$ will no longer be given by (38) but the highest derivatives in the new equation are identical with the left-hand side of (38). Since it is these which determine the characteristics, the situation illustrated in Fig. 4 will still occur and there will be a singularity at the point $r = 0$, $y = \eta_0 + b(-\eta_0^2)$, where η_0 is a root of (39).

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